
Minimal surfaces and harmonic diffeomorphisms from the complex plane onto a Hadamard surface.

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Abstract. We construct harmonic diffeomorphisms from the complex plane \mathbb{C} onto any Hadamard surface \mathbb{M} whose curvature is bounded above by a negative constant. For that, we prove a Jenkins-Serrin type theorem for minimal graphs in $\mathbb{M} \times \mathbb{R}$ over domains of \mathbb{M} bounded by ideal geodesic polygons and show the existence of a sequence of minimal graphs over polygonal domains converging to an entire minimal graph in $\mathbb{M} \times \mathbb{R}$ with the conformal structure of \mathbb{C} .

1 Introduction.

There are many harmonic diffeomorphisms from the complex plane \mathbb{C} onto the hyperbolic plane \mathbb{H} . They were constructed by finding entire minimal graphs in $\mathbb{H} \times \mathbb{R}$ whose conformal type is \mathbb{C} [CR]. The vertical projection of such a graph onto \mathbb{H} is such a harmonic diffeomorphism. It was conjectured that there was no such map [SY].

In this paper we will show there are harmonic diffeomorphisms from \mathbb{C} onto any Hadamard surface whose curvature is bounded above by a negative constant. The question of their existence was posed by R. Schoen.

We proceed as in [CR] by constructing entire minimal graphs in $\mathbb{M} \times \mathbb{R}$, of conformal type \mathbb{C} ; \mathbb{M} a complete simply connected Riemannian surface with curvature $K_{\mathbb{M}} \leq a < 0$. The construction of these graphs in $\mathbb{H} \times \mathbb{R}$ can be done in $\mathbb{M} \times \mathbb{R}$; the geometry of the asymptotic boundary of \mathbb{M} is sufficiently close to that of \mathbb{H} .

We are thus able to prove a Jenkins-Serrin type theorem for minimal graphs in $\mathbb{M} \times \mathbb{R}$, over domains of \mathbb{M} bounded by ideal geodesic polygons. There are several constructions in our paper, which we believe will be useful for future research.

An interesting question is whether our theorems hold when $K_{\mathbb{M}} < 0$.

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2 Preliminaries.

We will devote this Section to present some basic properties of Hadamard manifolds, which will be necessary for our study (see for instance [E1, E2] for details).

Let \mathbb{M} be a Hadamard manifold, that is, a complete simply connected Riemannian manifold with non positive sectional curvature. It is classically known that there is a unique geodesic joining two points of \mathbb{M} . Thus, the concept of (geodesic) convexity is naturally defined for sets in \mathbb{M} .

We say that two geodesics $\gamma_1(t), \gamma_2(t)$ of \mathbb{M} , parametrized by arc length, are asymptotic if there exists a constant $c > 0$ such that the distance $d(\gamma_1(t), \gamma_2(t))$ is less than c for all $t \geq 0$. Analogously, two unit vectors v_1, v_2 are said to be asymptotic if the corresponding geodesics $\gamma_{v_1}(t), \gamma_{v_2}(t)$ have this property.

To be asymptotic is an equivalence relation on the oriented unit speed geodesics or on the set of unit vectors of \mathbb{M} . Each one of these equivalence classes will be called a point at infinity, and $\mathbb{M}(\infty)$ will denote the set of points at infinity.

We will denote by $\gamma(+\infty)$ or $v(\infty)$ the equivalence class of the corresponding geodesic $\gamma(t)$ or unit vector v .

When \mathbb{M} is a Hadamard manifold with sectional curvature bounded from above by a negative constant then any two asymptotic geodesics γ_1, γ_2 satisfy that the distance between the two curves $\gamma_1|_{[t_0, +\infty)}, \gamma_2|_{[t_0, +\infty)}$ is zero for any $t_0 \in \mathbb{R}$. In addition, under this curvature hypothesis, given $x, y \in \mathbb{M}(\infty)$ there exists a unique oriented unit speed geodesic γ such that $\gamma(+\infty) = x$ and $\gamma(-\infty) = y$, where $\gamma(-\infty)$ is the corresponding point at infinity when we change the orientation of γ .

For any point p of a general Hadamard manifold, there is a bijective correspondence between the set of unit vectors at p and $\mathbb{M}(\infty)$, where a unit vector v is mapped to the point at infinity $v(\infty)$. Equivalently, given a point $p \in \mathbb{M}$ and a point $x \in \mathbb{M}(\infty)$, there exists a unique oriented unit speed geodesic γ such that $\gamma(0) = p$ and $\gamma(+\infty) = x$. In particular, $\mathbb{M}(\infty)$ is bijective to a sphere.

In fact, there exists a topology on $\mathbb{M}^* = \mathbb{M} \cup \mathbb{M}(\infty)$ satisfying

1. the restriction to \mathbb{M} agrees with the topology induced by the Riemannian distance,
2. there exists a homeomorphism from \mathbb{M}^* onto the closed unit ball which identifies $\mathbb{M}(\infty)$ with the unit sphere,
3. the map $v \rightarrow v(\infty)$ is a homeomorphism from the unit sphere of the tangent plane at a fixed point p onto $\mathbb{M}(\infty)$.

This topology is called the cone topology of \mathbb{M}^* and can be obtained as follows. Let $p \in \mathbb{M}$ and \mathcal{U} an open set in the unit sphere of its tangent plane. Define for any $r > 0$

$$T(\mathcal{U}, r) = \{\gamma_v(t) \in \mathbb{M}^* : v \in \mathcal{U}, r < t \leq +\infty\}.$$

The cone topology is the unique one such that its restriction to \mathbb{M} is the topology induced by the Riemannian distance and such that the sets $T(\mathcal{U}, r)$ containing a point $x \in \mathbb{M}(\infty)$ form a neighborhood basis at x .

Given a set $A \subseteq \mathbb{M}$, we denote by $\partial_\infty A$ the set $\partial A \cap \mathbb{M}(\infty)$, where ∂A is the boundary of A for the cone topology.

Horospheres are defined in terms of Busemann functions. Given a unit vector v , the Busemann function $B_v : \mathbb{M} \longrightarrow \mathbb{R}$, associated to v , is

$$B_v(p) = \lim_{t \rightarrow +\infty} d(p, \gamma_v(t)) - t.$$

This function verifies some important properties

1. B_v is a \mathcal{C}^2 convex function on \mathbb{M} ,
2. the gradient $\nabla B_v(p)$ is the unique unit vector w at p such that $v(\infty) = -w(\infty)$,
3. if w is a unit vector such that $v(\infty) = w(\infty)$ then $B_v - B_w$ is a constant function on \mathbb{M} .

Given a point $x \in \mathbb{M}(\infty)$ and a unit vector v such that $v(\infty) = x$ we define the horospheres at x as the level sets of the Busemann function B_v . By property 3, the horospheres at x do not depend on the choice of v . The horospheres at a point $x \in \mathbb{M}(\infty)$ give a foliation of \mathbb{M} and, from property one, each one bounds a convex domain in \mathbb{M} called a horoball. Moreover, the intersection between a geodesic γ and a horosphere at $\gamma(+\infty)$ is always orthogonal from property two.

With respect to distance from horospheres we present the following facts.

1. Let $p \in \mathbb{M}$, H_x a horosphere at x and γ the geodesic passing through p having x as a point at infinity, then $H_x \cap \gamma$ is the closest point on H_x to p .
2. If γ is a geodesic with points at infinity x, y , and H_x, H_y are disjoint horospheres at these points then the distance between H_x and H_y agrees with the distance between the points $H_x \cap \gamma$ and $H_y \cap \gamma$.
3. The function $\mathcal{D} : \mathbb{M} \times \mathbb{M}^* \times \mathbb{M} \longrightarrow \mathbb{R}$ given by

$$\mathcal{D}(a, b, c) = \begin{cases} d(c, b) - d(a, b) & \text{if } b \in \mathbb{M} \\ B_v(c) & \text{if } b \in \mathbb{M}(\infty) \end{cases} \quad (2.1)$$

is continuous, where v is the unique unit tangent vector at a such that $v(\infty) = b$. $\mathcal{D}(a, b, c)$ measures the difference between the oriented distance from a and c to any horosphere at $b \in \mathbb{M}(\infty)$. In particular, $\mathcal{D}(a, b, c) < 0$ means that c is in the horoball whose boundary is the horosphere at b passing across a .

3 A Jenkins-Serrin type theorem for ideal polygons.

From now on we will assume \mathbb{M} is a simply connected, complete surface with Gauss curvature bounded from above by a negative constant.

We say that Γ is an *ideal polygon* if Γ is a Jordan curve in \mathbb{M}^* which is a geodesic polygon with an even number of sides and all the vertices in $\mathbb{M}(\infty)$. As usual, we will denote by $A_1, B_1, \dots, A_k, B_k$ the sides of Γ , which are oriented counter-clockwise.

Now, we study the Dirichlet problem for the minimal surface equation in the domain D bounded by an ideal polygon Γ . That is, we look for a solution $u : D \rightarrow \mathbb{R}$ to the equation

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0. \quad (3.1)$$

Here, we prescribe the $+\infty$ data on each side A_i and $-\infty$ on each side B_i .

For relatively compact domains $D \subseteq \mathbb{M}$, it is well-known that there are necessary and sufficient conditions on the lengths of the sides of polygons inscribed in Γ in order to solve this Dirichlet problem (see [JS], [NR], [P]).

When Γ is an ideal polygon the length of each side is infinity and the previous conditions make no sense. However, in [CR], the authors devise a manner to compare the “lengths” of sides.

Fix an ideal polygon Γ and consider pairwise disjoint horocycles H_i at each vertex a_i of Γ .

For each side A_i , let us denote by \tilde{A}_i the compact geodesic arc between the horocycles at the vertices of A_i , and by $|\tilde{A}_i|$ the length of \tilde{A}_i , that is, the distance between the horocycles. Analogously, one defines \tilde{B}_i and $|\tilde{B}_i|$ for each side B_i , (cf. Figure 1).

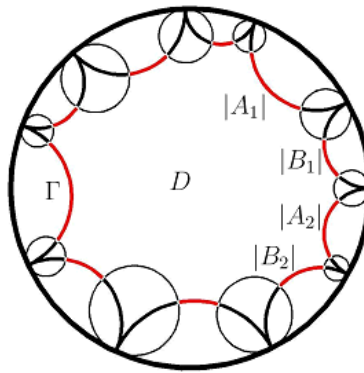


Figure 1.

Observe that if we define

$$a(\Gamma) = \sum_{i=1}^k |A_i|, \quad b(\Gamma) = \sum_{i=1}^k |B_i|,$$

then $a(\Gamma) - b(\Gamma)$ does not depend on the choice of horocycles. This is due to the fact that if we change a horocycle at a vertex then $a(\Gamma)$ and $b(\Gamma)$ increase or decrease in the same quantity.

Let D be the domain bounded by an ideal polygon Γ . We say that a simple closed geodesic polygon \mathcal{P} is *inscribed* in D if each vertex of \mathcal{P} is a vertex of Γ .

Each side of \mathcal{P} is one side A_i or B_i of Γ , or a geodesic contained in D (cf. Figure 2). Thus, the definition of $a(\Gamma)$ and $b(\Gamma)$ extends to \mathcal{P} . In addition, we define the *truncated length* of the inscribed polygon $|\mathcal{P}|$ as the sum of the lengths of the compact arcs of each side of \mathcal{P} bounded by the horocycles at its vertices.

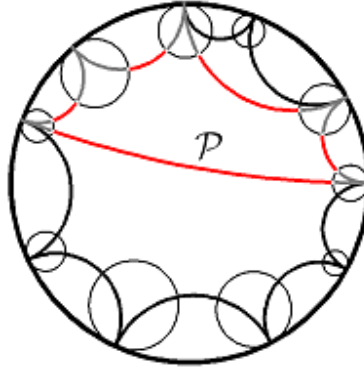


Figure 2.

Now, we can state a Jenkins-Serrin type theorem on domains of \mathbb{M} bounded by an ideal polygon Γ .

Theorem 3.1. *There is a solution to the Dirichlet problem for the minimal surface equation in the domain D bounded by Γ with prescribed data $+\infty$ at A_i and $-\infty$ at B_i if, and only if, the following two conditions are satisfied*

1. $a(\Gamma) - b(\Gamma) = 0$,
2. *for all inscribed polygons \mathcal{P} in D different from Γ there exist horocycles at the vertices such that*

$$2a(\mathcal{P}) < |\mathcal{P}| \quad \text{and} \quad 2b(\mathcal{P}) < |\mathcal{P}|.$$

Moreover, the solution is unique up to additive constants.

Remark 3.1. Notice that $a(\mathcal{P})$ and $b(\mathcal{P})$ depend on the chosen horocycles at the vertices. However, if condition 2 is satisfied for a particular choice of horocycles then it is also satisfied for all smaller horocycles at the vertices.

In addition, let A_i, B_j be the two sides of Γ with a common vertex of \mathcal{P} . If the side A_i does not belong to \mathcal{P} then $2a(\mathcal{P}) < |\mathcal{P}|$ is satisfied for the choice of a small horocycle at the vertex. Thus, if \mathcal{P} is an inscribed polygon in D such that there exists a vertex of \mathcal{P} not containing the adjacent side A_i of Γ and another vertex of \mathcal{P} not containing the adjacent side $B_{i'}$ of Γ , then condition 2 is satisfied for the polygon \mathcal{P} .

Proof of Theorem 3.1. This Theorem was proved in [CR] when \mathbb{M} is the hyperbolic plane \mathbb{H} . The reader can check their proof works for a general surface \mathbb{M} once the existence of a Scherk type surface on each halfspace of \mathbb{M} is established.

This Scherk type surface in \mathbb{H} is unique up to isometries of the ambient space and was explicitly computed by U. Abresch and R. Sa Earp [S]. For a general \mathbb{M} we now show its existence.

Proposition 3.1. Let γ be a complete geodesic in \mathbb{M} and Ω a connected component of $\mathbb{M} - \gamma$. There exists a positive solution u to the Dirichlet problem for the minimal surface equation in Ω with prescribed data $+\infty$ at γ and such that

$$\lim\{u(p_n)\} = 0$$

for each sequence $\{p_n\}$ of points in Ω with distance to γ going to infinity.

Proof. Since \mathbb{M} is a Hadamard surface then

$$\varphi(s, t) = \exp_{\gamma(t)}(s J\gamma'(t)), \quad (s, t) \in \mathbb{R}^2$$

is a global parametrization of \mathbb{M} , where the geodesic $\gamma(t)$ is parametrized by arc length, \exp is the usual exponential map and J stands for the rotation in \mathbb{M} by $\pi/2$. In addition, we can assume that Ω is parametrized for $s > 0$.

We observe that

$$\left\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right\rangle = 1, \quad \left\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right\rangle = 0$$

where \langle, \rangle is the induced metric in \mathbb{M} . Moreover, if we denote by $G(s, t)$ the function $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle$ then

$$G(0, t) = 1, \quad G_s(0, t) = 0, \quad t \in \mathbb{R} \quad (3.2)$$

since $\gamma(t)$ is a geodesic. Here, G_s denotes the derivative of G with respect to s .

Now, we consider a graph $\psi(s, t) = (\varphi(s, t), h(s))$ on Ω which has constant height for equidistant points to γ , that is, when s is constant.

For the unit normal of the graph pointing down, the mean curvature of the immersion is positive if and only if

$$G_s h_s(1 + h_s^2) + 2G h_{ss} < 0, \quad s > 0, t \in \mathbb{R}. \quad (3.3)$$

On the other hand, the Gauss curvature of \mathbb{M} is given by

$$K(s, t) = -\frac{1}{4} \left(\frac{G_s}{G} \right)^2 - \frac{1}{2} \left(\frac{G_s}{G} \right)_s. \quad (3.4)$$

We notice that, for any constant $d < 0$, the function $\tilde{G}(s) = \cosh^2(\sqrt{-d}s)$ verifies

$$d = -\frac{1}{4} \left(\frac{\tilde{G}_s}{\tilde{G}} \right)^2 - \frac{1}{2} \left(\frac{\tilde{G}_s}{\tilde{G}} \right)_s. \quad (3.5)$$

Observe that $ds^2 + \tilde{G}(s) dt^2$ is the hyperbolic metric of curvature d . Moreover, the function

$$\tilde{h}(s) = -\frac{1}{\sqrt{-d}} \log \left(\tanh \left(\frac{1}{2} \sqrt{-d} s \right) \right), \quad s > 0, \quad (3.6)$$

is decreasing and satisfies

$$\tilde{G}_s \tilde{h}_s(1 + \tilde{h}_s^2) + 2\tilde{G} \tilde{h}_{ss} = 0. \quad (3.7)$$

That is, $\tilde{h}(s)$ is the minimal graph in the hyperbolic space found by Abresch and Sa Earp.

Since $K(s, t)$ is bounded from above by a negative constant c , we can choose d such that $c < d < 0$. Then from (3.4) and (3.5)

$$\left(\frac{G_s}{G} \right)^2 + 2 \left(\frac{G_s}{G} \right)_s > \left(\frac{\tilde{G}_s}{\tilde{G}} \right)^2 + 2 \left(\frac{\tilde{G}_s}{\tilde{G}} \right)_s. \quad (3.8)$$

Now, we observe that given two real functions $f(x), g(x)$ defined on an interval I , with $f(x_0) = g(x_0)$ and satisfying

$$2f'(x) + f(x)^2 > 2g'(x) + g(x)^2,$$

then $f(x) > g(x)$, for all $x > x_0$ on I .

Thus, from (3.2), (3.7) and (3.8),

$$\frac{G_s}{G} > \frac{\tilde{G}_s}{\tilde{G}} = \frac{-2\tilde{h}_{ss}}{\tilde{h}_s(1 + \tilde{h}_s^2)}, \quad s > 0, t \in \mathbb{R}$$

or equivalently, $h = \tilde{h}$ satisfies the inequality in (3.3). That is, the graph $\psi(s, t) = (\varphi(s, t), \tilde{h}(s))$ on Ω has strictly positive mean curvature for its unit normal pointing down. This graph has value $+\infty$ on γ and goes to zero when the distant to γ tends to infinity.

Finally, we obtain the minimal graph with the desired properties as follows. Let $p \in \gamma \subseteq \mathbb{M}$, $C(n)$ the geodesic circumference in \mathbb{M} centered at p and radius n , $A(n) = C(n) \cap \Omega$ and $B(n)$ the segment $\gamma([-n, n])$. Now, consider the Jordan curve $\Gamma(n)$ in $\mathbb{M} \times \mathbb{R}$ obtained by the arcs $A(n) \times \{0\}$, $B(n) \times \{n\}$ and the vertical segments joining their end points. Let Σ_n be the minimal disk which is a solution of the Plateau problem for $\Gamma(n)$.

Σ_n is the graph of a function u_n on the domain bounded by $A(n) \cup B(n)$ with $u_n|_{A(n)} = 0$ and $u_n|_{B(n)} = n$. The sequence $\{u_n\}$ is non decreasing and non negative. In addition, from the comparison principle, it is bounded from above by \tilde{h} . Therefore, Σ_n converges to a minimal graph on Ω . This completes the proof of Proposition 3.1; hence, Theorem 3.1 as well. \square

As in [CR], we can extend Theorem 3.1 to more general domains. We say that a convex domain $D \subseteq \mathbb{M}$ is *admissible* if

1. the (non empty) finite set $\partial_\infty D$ are the vertices of an ideal polygon,
2. given two convex arcs $C_1, C_2 \subseteq \partial D$ with a common vertex $x \in \partial_\infty D$ there exist two sequences of points $x_n \in C_1, y_n \in C_2$ converging to x , such that the distance between x_n and y_n tends to zero.

The second condition in the above definition is used in order to obtain a maximum principle for minimal graphs over the domain D . Moreover, the domain bounded by an ideal polygon is admissible since the distance between two geodesics with a common point at infinity goes to zero.

Now, we present a Jenkins-Serrin type theorem when we fix continuous boundary data on some components of the boundary, whose proof can be shown as in [CR].

Let D be an admissible domain in \mathbb{M} . We seek a solution to the Dirichlet problem for the minimal surface equation in D which is $+\infty$ on geodesic sides A_1, \dots, A_k of ∂D , and equals $-\infty$ on other geodesic sides $B_1, \dots, B_{k'}$ of ∂D , and equal to continuous functions $f_i : C_i \rightarrow \mathbb{R}$ on the remaining (nonempty) convex arcs of ∂D .

Theorem 3.2. *There exists a unique solution in D to the above Dirichlet problem if, and only if,*

$$2a(\mathcal{P}) < |\mathcal{P}| \quad \text{and} \quad 2b(\mathcal{P}) < |\mathcal{P}|$$

for all inscribed polygon \mathcal{P} in D .

The uniqueness property in the two previous Theorems is guaranteed by a maximum principle over admissible domains. The proof when $\mathbb{M} = \mathbb{H}$ was given in [CR] and it also works in our situation.

Theorem 3.3. (Generalized Maximum Principle) *Let $D \subseteq \mathbb{M}$ be an admissible domain. Let us consider a domain $\Omega \subseteq D$ and $u, v \in C^0(\overline{\Omega})$ two solutions to the minimal surface equation in Ω with $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω .*

In addition, as it was explained in [CR], the previous maximum principle also applies when the solutions u, v to the minimal surface equation have infinite boundary values along some geodesic arcs of ∂D . When ∂D is made of complete geodesics and u, v take the same infinite value on the whole boundary of D , then u and v agree up to an additive constant.

4 Extending the domain of a solution.

The aim of this section is to show that a solution u to the minimal surface equation on an admissible polygonal domain D with infinity boundary data can be “extended” to a larger polygonal domain. By this we mean that a solution v on a larger domain can be chosen arbitrarily close to u on a fixed compact set K of D .

To establish this result, we first need to show the existence of some special ideal quadrilaterals. We will call *ideal Scherk surface* a graph given by Theorem 3.1, and *Scherk domain* the domain where it is defined.

Proposition 4.1. *Let x, y, z be three points in $\mathbb{M}(\infty)$. Let γ be the geodesic joining x and y , and Ω the connected component of $\mathbb{M} - \gamma$ such that $z \notin \partial_\infty \Omega$. Then there exist a point $w \in \partial_\infty \Omega$ and an ideal Scherk surface over the domain bounded by the ideal quadrilateral with vertices x, y, z, w . (Cf. Figure 3.)*

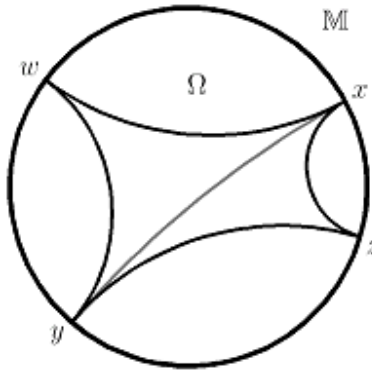


Figure 3.

We start by establishing some previous Lemmas in order to prove the above Proposition.

Lemma 4.1. *Let H_1, H_2 be two different horocycles in \mathbb{M} . Then they intersect at most at two points.*

Proof. Let us assume that H_1, H_2 are horocycles at $x, y \in \mathbb{M}(\infty)$, respectively. If $x = y$ then they do not intersect. So, we can suppose $x \neq y$.

Let $p \in H_1 \cap H_2$. Then, the intersection between the two horocycles at p is transversal, unless p is in the geodesic γ_{xy} joining x and y . In the latter case, if γ_p is the geodesic through p tangent to H_1 and H_2 , then each convex horodisk B_1, B_2 , with respective boundaries H_1, H_2 , must be on different sides of γ_p . Therefore, each horocycle is in the concave part of the other and the intersection is only p .

Thus, if $H_1 \cap H_2$ has more than one point then each intersection is transversal. Let us consider $p_1, p_2 \in H_1 \cap H_2$ and \mathcal{I} the compact arc in H_1 joining p_1 and p_2 . Let p_0 be a point in \mathcal{I} at the largest distance from H_2 . Then, the horocycle \widetilde{H}_2 at y passing through p_0 intersects H_1 in a tangent way. In particular, p_0 is the point $\gamma_{xy} \cap H_1$.

So, for any two points $p_1, p_2 \in H_1 \cap H_2$, we have that p_0 is in the interior of the compact arc of H_1 joining p_1 and p_2 . Therefore, there are at most two points in the intersection between H_1 and H_2 . \square

Lemma 4.2. *Let $x, y \in \mathbb{M}(\infty)$, and H_x, H_y two disjoint horocycles at x, y , respectively. Consider the open set \mathcal{I} of points in $\mathbb{M}(\infty)$ between x and y , where we assume $\mathbb{M}(\infty)$ ordered counter-clockwise. For any point z , we define $L : \mathcal{I} \rightarrow \mathbb{R}$ as*

$$L(z) = d(H_y, H_z) - d(H_z, H_x),$$

where H_z is any horocycle at z disjoint from the previous ones and $d(\cdot, \cdot)$ denotes distance in \mathbb{M} . Then, L is a homeomorphism from \mathcal{I} onto \mathbb{R} .

Proof. We observe that the definition of the function L does not depend on the chosen horocycle H_z at z , and also makes sense when H_z intersects H_x or H_y in one point.

Let us consider the homeomorphism $h_1 : \mathbb{M}(\infty) - \{x\} \rightarrow H_x$ sending each point $z \in \mathbb{M}(\infty)$ different from x to the intersection between H_x and the geodesic joining x and z . Analogously, we consider the homeomorphism $h_2 : \mathbb{M}(\infty) - \{y\} \rightarrow H_y$.

Then $L(z) = d(h_2(z), H_z) - d(H_z, h_1(z)) = \mathcal{D}(h_1(z), z, h_2(z))$ is a continuous function, where \mathcal{D} is given by (2.1).

Now, we see that L is injective. Let p, q be two points in \mathcal{I} oriented such that $x < p < q < y$. Let H_p, H_q be the smallest horocycles at p, q , respectively, such that they intersect $H_x \cup H_y$. We distinguish three cases:

1. H_q only intersects H_y and H_p only intersects H_x ,
2. H_q intersects H_x ,
3. H_p intersects H_y .

The case 1 is trivial because $L(q) < 0 < L(p)$. And, since the cases 2 and 3 are symmetric, we only need to study case 2.

First, we observe that H_p does not intersect H_y . Otherwise, the intersection holds at $h_2(p)$ and each connected component of $H_p - \{h_2(p)\}$ intersects twice to H_q , which contradicts Lemma 4.1 (cf. Figure 4).

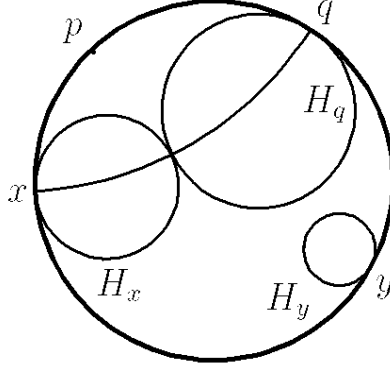


Figure 4.

Thus, H_p and H_q intersect H_x and we only need to show that $d(H_q, H_y) < d(H_p, H_y)$ in order to obtain that $L(q) < L(p)$. For that, enlarge H_y to the first horocycle \widetilde{H}_y that intersects $H_p \cup H_q$. From the previous discussion, replacing H_y and \widetilde{H}_y , H_p does not intersect \widetilde{H}_y (cf. Figure 5). Hence, $d(H_q, H_y) < d(H_p, H_y)$ and the case 2 is proved.

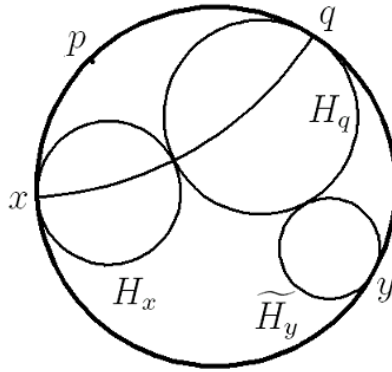


Figure 5.

Now, let us see that there exists a sequence of points $p_n \in \mathcal{I}$ such that $L(p_n)$ goes to $-\infty$.

Let γ be the geodesic joining x and y and Ω the connected component of $\mathbb{M} - \gamma$ which contains \mathcal{I} in its ideal boundary. Consider $q \in H_y \cap \Omega$ and γ_q the unique geodesic joining q and x . Let us denote by p the other point of γ_q at the ideal boundary.

The smallest horocycle H_p intersecting $H_x \cup H_y$ does not intersect H_x . Otherwise, γ_q would be contained in the horodisks bounded by H_x and H_q . But, since $q \in \gamma_q \cap H_y$ then q would be in the horodisk bounded by H_p , which is a contradiction (cf. Figure 6).

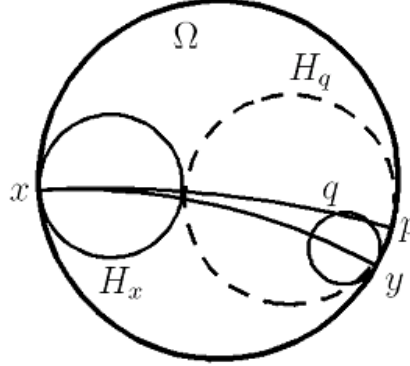


Figure 6.

Thus $L(p) = -d(H_p, H_x) = -(d(H_p, q) + d(q, H_x)) \leq -d(q, H_x)$. Therefore, taking a sequence $q_n \in H_y \cap \Omega$ converging to y , we obtain the corresponding sequence p_n such that $\lim L(p_n) \leq -\lim d(q_n, H_x) = -\infty$.

Analogously, it can be proved that there exists a sequence of points $p_n \in \mathcal{I}$ such that $L(p_n)$ goes to $+\infty$.

Hence, since \mathcal{I} has the topology of an interval, L is a strictly monotonous continuous function, and there exist sequences in \mathcal{I} whose image tend to $-\infty$ and $+\infty$, then L is a homeomorphism from \mathcal{I} onto \mathbb{R} . \square

Now, let us denote by $|xy|$ the distance between two points in $\mathbb{M}^* = \mathbb{M} \cup \mathbb{M}(\infty)$, where we indicate distance between horocycles if x or y are in $\mathbb{M}(\infty)$.

Lemma 4.3. (Generalized Triangle Inequality.) *Consider a triangle with vertices x_1, x_2 in $\mathbb{M}^* = \mathbb{M} \cup \mathbb{M}(\infty)$ and another point $x_3 \in \mathbb{M}$. Then,*

$$|x_1x_2| \leq |x_1x_3| + |x_3x_2|.$$

Moreover, if $x_1, x_2, x_3 \in \mathbb{M}(\infty)$ then there exist horocycles at these points such that the following three inequalities are simultaneously satisfied

$$|x_ix_j| < |x_ix_k| + |x_kx_j|, \quad \{i, j, k\} = \{1, 2, 3\}.$$

Remark 4.1. *When $x_3 \in \mathbb{M}$, the quantity $|x_1x_3| + |x_3x_2| - |x_1x_2|$ does not depend on the chosen disjoint horocycles, if any. However, it is important to bear in mind that $|x_ix_k| + |x_kx_j| - |x_ix_j|$ depends on the chosen horocycles if $x_1, x_2, x_3 \in \mathbb{M}(\infty)$.*

Proof of Lemma 4.3. First, we consider the case $x_3 \in \mathbb{M}$.

If $x_1, x_2 \in \mathbb{M}$ then the inequality is clear. Thus, let us assume $x_1 \in \mathbb{M}(\infty)$. Then, enlarge the horocycle H_{x_1} to another horocycle \widetilde{H}_{x_1} which intersects x_3 or x_2 (or H_{x_2}), for the first time.

If x_2 (or H_{x_2}) intersects \widetilde{H}_{x_1} the inequality is clear. Otherwise, $x_3 \in \widetilde{H}_{x_1}$, and so the distance from x_2 (or H_{x_2}) to \widetilde{H}_{x_1} is less than or equal to the distance to x_3 and the inequality also holds.

Finally, we consider $x_1, x_2, x_3 \in \mathbb{M}(\infty)$ and three pairwise disjoint horocycles $H_{x_1}, H_{x_2}, H_{x_3}$. Now, fix H_{x_1}, H_{x_2} and consider a small enough horocycle \widetilde{H}_{x_3} such that $|x_1x_2| < |x_1x_3| + |x_3x_2|$.

To obtain the second inequality, we consider a smaller horocycle \widetilde{H}_{x_2} at x_2 , if necessary, such that $|x_1x_3| < |x_1x_2| + |x_2x_3|$. And now we observe that the first inequality remains unchanged for the horocycles $H_{x_1}, \widetilde{H}_{x_2}, \widetilde{H}_{x_3}$.

Following the same process, we take a small horocycle \widetilde{H}_{x_1} at x_1 such that $|x_2x_3| < |x_2x_1| + |x_1x_3|$. Since the previous two inequalities do not change, the Lemma follows. \square

Proof of Proposition 4.1. Let H_z be a horocycle at z . Take H_x, H_y disjoint horocycles at x, y , respectively, at the same distance from H_z . Then, using Lemma 4.2, there exists a point $w \in \partial_\infty \Omega$ such that $d(H_w, H_x) = d(H_w, H_y)$ for any horocycle H_w at w disjoint from H_x and H_y . That is, $a(\Gamma) - b(\Gamma) = 0$ for the ideal quadrilateral Γ with vertices x, y, z, w .

Finally, from the Generalized Triangle Inequality, condition 2 in Theorem 3.1 is satisfied and, so, there exists an ideal Scherk surface over the domain bounded by Γ . \square

Now, we establish some notation.

Given an even set of points $a_0, a_1, \dots, a_{2n-1}$ in $\mathbb{M}(\infty)$, which we will assume ordered counter-clockwise, we denote by $\mathcal{P}(a_0, a_1, \dots, a_{2n-1})$ the ideal polygon in \mathbb{M} whose vertices are these points.

In order to obtain an ideal Scherk surface on the domain bounded by $\mathcal{P}(a_0, a_1, \dots, a_{2n-1})$, we will fix $+\infty$ boundary data on the sides $[a_{2k}, a_{2k+1}]$ and $-\infty$ boundary data on the sides $[a_{2k+1}, a_{2k+2}]$. Here, $[x, y]$ denotes the complete geodesic joining the points $x, y \in \mathbb{M}(\infty)$, and we identify $a_{2n} = a_0$.

Proposition 4.2. *Let u be an ideal Scherk graph on the domain D bounded by an ideal polygon $\mathcal{P}(a_0, a_1, a_2, \dots, a_{2n-1})$. Now, we attach to D two Scherk domains bounded by $\mathcal{P}(a_0, b_1, b_2, a_1)$ and $\mathcal{P}(a_1, b_3, b_4, a_2)$. Then, given a compact set $K \subseteq D$ and $\varepsilon > 0$, there exists an ideal Scherk graph v on the domain bounded by the ideal polygon $\mathcal{P}(a_0, b_1, b'_2, a_1, b'_3, b_4, a_2, \dots, a_{2n-1})$ such that*

$$\|v - u\|_{C^2(K)} \leq \varepsilon$$

where b'_2, b'_3 can be chosen in any punctured neighborhood of b_2, b_3 in $\mathbb{M}(\infty)$, respectively. (Cf. Figure 7.)

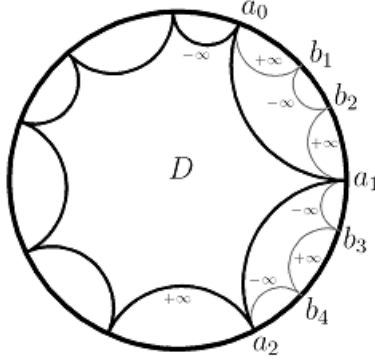


Figure 7.

Remark 4.2. We observe that the existence of the Scherk domains bounded by $\mathcal{P}(a_0, b_1, b_2, a_1)$ and $\mathcal{P}(a_1, b_2, b_3, a_2)$ is guaranteed by Proposition 4.1.

The proof of Proposition 4.2 proceeds as in [CR]; we will first prove three lemmas. Following the notation in Proposition 4.2, we denote by E_1 the domain bounded by the polygon $\mathcal{P}(a_0, b_1, b_2, a_1)$, by E_2 the domain bounded by $\mathcal{P}(a_1, b_3, b_4, a_2)$ and by D_0 the global domain bounded by $\Gamma = \mathcal{P}(a_0, b_1, b_2, a_1, b_3, b_4, a_2, \dots, a_{2n-1})$. Then, it is clear that Γ satisfies Condition 1 in Theorem 3.1 and, in addition, one obtains

Lemma 4.4. Condition 2 in Theorem 3.1 is satisfied by every inscribed polygon in D_0 , except the boundaries of $E_1, E_2, D_0 - E_1$ and $D_0 - E_2$.

Proof. It is clear that the boundaries of $E_1, E_2, D_0 - E_1$ and $D_0 - E_2$ do not satisfy Condition 2 in Theorem 3.1. Therefore, we start with a inscribed polygon \mathcal{P} in D_0 different from them.

From Remark 3.1, we can assume that \mathcal{P} has the adjacent side of ∂D_0 with $+\infty$ data, at any vertex of \mathcal{P} . And we only have to prove that $2a(\mathcal{P}) < |\mathcal{P}|$, for some choice of horocycles at the vertices.

Let $D_{\mathcal{P}}$ be the domain bounded by \mathcal{P} and \mathcal{P}' the boundary of $D_{\mathcal{P}} - E_2$. \mathcal{P}' is a polygon with some possible vertices in \mathbb{M} (cf. Figure 8). We are going to show that if $2a(\mathcal{P}') < |\mathcal{P}'|$ then $2a(\mathcal{P}) < |\mathcal{P}|$. And, so, we will only need to prove that the inequality is true for \mathcal{P}' .

If the geodesic $[b_3, b_4]$ does not belong to \mathcal{P} then $\mathcal{P} = \mathcal{P}'$ and the result is obvious. Let d_1 be the vertex of \mathcal{P} previous to b_3 , and d_2 the vertex following b_4 . Consider $q_1 = [d_1, b_3] \cap [a_1, a_2]$ and $q_2 = [b_4, d_2] \cap [a_1, a_2]$ (cf. Figure 8). Observe that q_i could be a_i ; in that case we have $|a_i q_i| = 0$. We have,

$$\begin{aligned} a(\mathcal{P}) &= a(\mathcal{P}') + |b_3 b_4|, \\ |\mathcal{P}| &= |\mathcal{P}'| - |q_1 q_2| + |q_1 b_3| + |b_3 b_4| + |b_4 q_2|. \end{aligned}$$

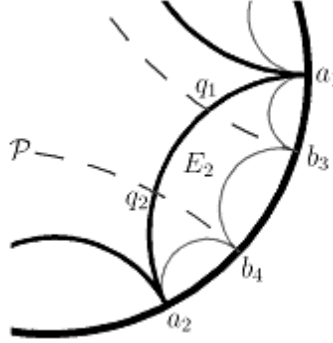


Figure 8.

On the other hand, since E_2 is a Scherk domain we can assume $d_0 = |a_1 b_3| = |b_3 b_4| = |b_4 a_2| = |a_1 a_2|$. Thus, from the Generalized Triangle Inequality, $|a_1 b_3| \leq |a_1 q_1| + |q_1 b_3|$, $|a_2 b_4| \leq |a_2 q_2| + |q_2 b_4|$, and using that $2a(\mathcal{P}') < |\mathcal{P}'|$, we have

$$\begin{aligned} |\mathcal{P}| - 2a(\mathcal{P}) &> |q_1 b_3| + |b_4 q_2| - |q_1 q_2| - |b_3 b_4| \\ &= |q_1 b_3| + |b_4 q_2| - 2|b_3 b_4| + |a_1 q_1| + |a_2 q_2| \\ &= (|q_1 b_3| + |a_1 q_1| - |a_1 b_3|) + (|b_4 q_2| + |a_2 q_2| - |a_2 b_4|) \geq 0. \end{aligned}$$

Therefore, in order to finish the proof of Lemma 4.4 we only need to see that $2a(\mathcal{P}') < |\mathcal{P}'|$.

Let $D_{\mathcal{P}'}$ be the domain bounded by \mathcal{P}' , and \mathcal{P}'' the boundary of $D_{\mathcal{P}'} - E_1$. We use a flux inequality, for the initial minimal graph u , over the domain bounded by \mathcal{P}'' to obtain the desired inequality.

From the minimal graph equation (3.1), the field $X = (\nabla u)/W$, with $W = \sqrt{1 + |\nabla u|^2}$ is divergence free. We write \mathcal{P}'' as the union of three sets: the union of all geodesic arcs of \mathcal{P}'' with boundary data $+\infty$ and disjoint from $[a_0, a_1]$, $I_1 = \mathcal{P}'' \cap [a_0, a_1]$ and J the union of the remaining arcs.

If ν is the unit outward normal to \mathcal{P}'' then $X = \nu$ on the sides with $+\infty$ boundary data (see [CR, P]), and so

$$0 = F_u(\partial \mathcal{P}'') = a(\mathcal{P}'') + |I_1| + F_u(J) + \rho,$$

where, for instance, $F_u(\partial \mathcal{P}'') = \int_{\partial \mathcal{P}''} \langle X, \nu \rangle$ denotes the flux of u along $\partial \mathcal{P}''$.

Here, the flux $F_u(J)$ is taken on the compact arcs of J outside the horocycles, and the number ρ corresponds to the remaining flux of X along some parts of horocycles.

Since $|\mathcal{P}''| = a(\mathcal{P}'') + |I_1| + |J|$, we have

$$|\mathcal{P}''| - 2(a(\mathcal{P}'') + |I_1|) = |J| + F_u(J) + \rho. \quad (4.1)$$

We are assuming that \mathcal{P} has the adjacent side of ∂D_0 with $+\infty$ data, for any vertex of \mathcal{P} . Therefore, the estimation of $|\mathcal{P}'| - 2a(\mathcal{P}')$ does not depend on the chosen horocycles. Moreover,

\mathcal{P}'' is not empty and it is different from the boundary of D . To see that, observe that if \mathcal{P}'' is empty then the previous condition on \mathcal{P} implies that \mathcal{P} is the boundary of E_1 , and if \mathcal{P}'' is the boundary of D then \mathcal{P} is the boundary of $D \cup E$ or D_0 .

Hence, \mathcal{P}'' has interior arcs in D , and $F_u(\alpha) + |\alpha|$ is positive on each interior arc α . Thus, $|J| + F_u(J)$ is positive and non decreasing when we choose smaller horocycles. In addition, we can select these horocycles so that $|\rho|$ is as small as desired. Therefore, from (4.1), we can assume

$$|\mathcal{P}''| - 2(a(\mathcal{P}'') + |I_1|) > 0 \quad (4.2)$$

for suitable horocycles at the vertices.

Now, we show that $2a(\mathcal{P}') < |\mathcal{P}'|$. For that, we distinguish three cases

1. a_0 and a_1 are vertices of \mathcal{P} ,
2. only one of the points a_0, a_1 is a vertex of \mathcal{P} ,
3. neither a_0 nor a_1 are vertices of \mathcal{P} .

In the case 1, $[a_0, b_1]$ and $[b_2, a_1]$ must be sides of \mathcal{P} . So, $[b_1, b_2]$ is also a side of \mathcal{P} and E_1 is contained in the domain bounded by \mathcal{P} . Thus,

$$a(\mathcal{P}') = a(\mathcal{P}'') + 2d_0, \quad |\mathcal{P}'| = |\mathcal{P}''| + 2d_0 \quad \text{and} \quad |I_1| = d_0.$$

And (4.2) gives us $2a(\mathcal{P}') < |\mathcal{P}'|$.

For the case 2, we assume for instance that a_0 is a vertex of \mathcal{P} , but not a_1 . Then b_2 is not a vertex of \mathcal{P} and we consider the point q which is the intersection between the geodesic $[a_0, a_1]$ and the geodesic joining b_1 and the following vertex of \mathcal{P} (cf. Figure 9). Then $I_1 = [a_0, q]$ and

$$a(\mathcal{P}') = a(\mathcal{P}'') + d_0, \quad |\mathcal{P}'| = |\mathcal{P}''| - |I_1| + d_0 + |b_1q|.$$

And using (4.2)

$$\begin{aligned} 0 &< (|\mathcal{P}'| + |I_1| - d_0 - |b_1q|) - 2(a(\mathcal{P}') - d_0 + |I_1|) \\ &= |\mathcal{P}'| - 2a(\mathcal{P}') + d_0 - |I_1| - |b_1q|. \end{aligned}$$

Therefore, using the Generalized Triangle Inequality for the triangle with vertices a_0, b_1, q we have $d_0 - |I_1| - |b_1q| \leq 0$ and the inequality $2a(\mathcal{P}') < |\mathcal{P}'|$ is proved.

Case 3 is clear because the domain bounded by \mathcal{P}' lies on D and the flux formula (4.2) gives the desired inequality. \square

Now we will perturb E_1 and E_2 to obtain a Scherk domain. We will do this so that E_1, E_2 and their complements are inscribed polygons satisfying Condition 2 of Theorem 3.1. By the previous Lemma 4.4, the other inscribed polygons in D_0 also satisfy Condition 2. Hence, if we make the perturbation of E_1, E_2 small enough, the strict inequalities (Condition 2) satisfied by these inscribed polygons will remain strict inequalities. We now make this precise.

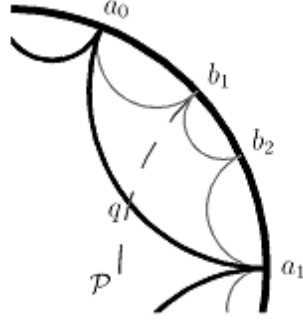


Figure 9.

Lemma 4.5. *For any punctured neighborhoods of b_2, b_3 in $\mathbb{M}(\infty)$, we can choose, respectively, two points b'_2, b'_3 such that the domain bounded by $\mathcal{P}(a_0, b_1, b'_2, a_1, b'_3, b_4, a_2, \dots, a_{2n-1})$ is, in fact, a Scherk domain.*

Proof. We first observe that

$$\begin{aligned} |a_0 a_1| - |a_1 b_2| + |b_2 b_1| - |b_1 a_0| &= 0 \\ |a_2 a_1| - |a_1 b_3| + |b_3 b_4| - |b_4 a_2| &= 0 \end{aligned}$$

since E_1 and E_2 are Scherk domains.

Now, using Lemma 4.2, there exist unique $b_2(t), b_3(t)$ such that

$$\begin{aligned} t &= |a_0 a_1| - |a_1 b_2(t)| + |b_2(t) b_1| - |b_1 a_0| \\ &= |a_2 a_1| - |a_1 b_3(t)| + |b_3(t) b_4| - |b_4 a_2|, \end{aligned} \tag{4.3}$$

where $b_2(t)$ varies in the open interval between b_1 and a_1 at infinity, $b_3(t)$ between a_1 and b_4 , and $t \in \mathbb{R}$. In addition, the functions $b_i(t)$ are homeomorphisms.

Let $\Gamma(t) = \mathcal{P}(a_0, b_1, b_2(t), a_1, b_3(t), b_4, a_2, \dots, a_{2n-1})$. From (4.3), Condition 1 in Theorem 3.1 is satisfied for any $t \in \mathbb{R}$. In addition, if $t > 0$ then the domains $E_1(t)$ and $E_2(t)$ bounded by $\mathcal{P}(a_0, b_1, b_2(t), a_1)$, $\mathcal{P}(a_1, b_3(t), b_4, a_2)$ and their complements also satisfy Condition 2. In order to obtain Condition 2 for the other inscribed polygons we argue as follows.

From Lemma 4.4, every polygon inscribed in the domain bounded by $\Gamma(0)$ satisfies Condition 2, except $E_1(0)$, $E_2(0)$ and their complements. Observe that the inequalities in Condition 2 are strict, and the number of inscribed polygons is finite. From Lemma 4.2, these inequalities depend continuously on $b_2(t)$ and $b_3(t)$, so one has that there exists $t_0 > 0$ such that Condition 2 is also satisfied for any domain bounded by $\Gamma(t)$, with $0 < t < t_0$. \square

Proof of Proposition 4.2. The proof is a verification that the arguments in [CR] work in our context. Let us denote by D_t the Scherk domain bounded by $\Gamma(t)$. Consider the graph of a

Scherk surface u_t defined on D_t with the corresponding infinite boundary data. First, we show that ∇u is the limit of $\nabla u_t|_D$ when t goes to zero.

Let us consider the divergence free fields $X_t = (\nabla u_t)/W_t$ and $X = (\nabla u)/W$, with $W_t = \sqrt{1 + |\nabla u_t|^2}$, $W = \sqrt{1 + |\nabla u|^2}$, associated to u_t and u , respectively. We now see that X_t converges to X on D when t tends to zero.

Consider the outer pointing normal ν along the boundary of D . We have fixed the same infinite boundary data on $\partial D - ([a_0, a_1] \cup [a_1, a_2])$, so $X_t = X = \pm \nu$ on this set.

On the boundary of $E_1(t)$ truncated by the horocycles, the flux of X_t is zero. Hence,

$$0 = |a_0 b_1| - |b_1 b_2(t)| + |b_2(t) a_1| + \int_{[a'_0, a'_1]} \langle X_t, -\nu \rangle + F_{u_t}(I_t),$$

where $[a'_0, a'_1]$ is the compact geodesic arc between the horocycles at a_0 and a_1 , and I_t is the set of arcs included in the four horodisks. Then, from (4.3),

$$t = \int_{[a'_0, a'_1]} (1 - \langle X_t, \nu \rangle) + F_{u_t}(I_t),$$

and taking limits for smaller horocycles at the vertices one has the convergence of the integral on the whole geodesic and

$$t = \int_{[a_0, a_1]} (1 - \langle X_t, \nu \rangle) = \int_{[a_0, a_1]} \langle X - X_t, \nu \rangle,$$

since $X = \nu$ on $[a_0, a_1]$.

Analogously, one has

$$t = - \int_{[a_1, a_2]} \langle X - X_t, \nu \rangle.$$

Thus, for any family α of disjoint arcs of ∂D

$$\left| \int_{\alpha} \langle X - X_t, \nu \rangle \right| \leq \int_{[a_0, a_1] \cup [a_1, a_2]} |\langle X - X_t, \nu \rangle| = 2t. \quad (4.4)$$

Now, we study the behavior of the field $X - X_t$ on the interior of D . Let Σ be the graph of u and Σ_t the graph of u_t . These graphs are stable, complete and satisfy uniform curvature estimates by Schoens' curvature estimates. Thus,

$$\forall \varepsilon > 0 \quad \exists \rho > 0 \text{ such that } \forall p \in D \quad \forall q \in \Sigma_t \cap B((p, u_t(p)), \rho) \text{ one has } \|N_t(p) - N_t(q)\| \leq \varepsilon.$$

Here, ρ does not depend on t , and N_t denotes the normal to Σ_t pointing down and $B((p, v_t(p)), \rho)$ the ball of radius ρ , centered at $(p, v_t(p)) \in \mathbb{M} \times \mathbb{R}$. These estimates remain true for Σ .

Therefore, one obtains that fixed $\varepsilon > 0$ and $p \in D$ there exists $\rho_1 \leq \rho/2$, which depends continuously on p but does not depend on t , such that for every q in the disk $B(p, \rho_1)$ in \mathbb{M} with center p and radius ρ_1 , we have $|u(q) - u(p)| \leq \rho/2$.

Let us assume now that $\|N_t(p) - N(p)\| \geq 3\varepsilon$. Consider the connected component $\Omega_t(p)$ of $\{q \in D : u(q) - u_t(q) > u(p) - u_t(p)\}$ with p in its boundary, and Λ_t the component of $\partial\Omega_t(p)$ containing p . Since Λ_t is a level curve of $u - u_t$ then it is piecewise smooth. Let $\sigma \subseteq \Sigma$, $\sigma_t \subseteq \Sigma_t$ be the two parallel curves which project on $\Lambda_t \cap B(p, \rho_1)$.

For the points of σ , we have that if $q \in \Lambda_t \cap B(p, \rho_1)$ then $|(q, u(q)) - (p, u(p))| \leq \rho_1 + \rho/2 \leq \rho$ and so $\|N(q) - N(p)\| \leq \varepsilon$. The same is also true on the parallel curve σ_t , that is, $\|N_t(q) - N_t(p)\| \leq \varepsilon$ for all $q \in \Lambda_t \cap B(p, \rho_1)$.

Thus, using these inequalities and the assumption on the normals at p , we obtain for all $q \in \Lambda_t \cap B(p, \rho_1)$ that $\|N(q) - N_t(q)\| \geq \|N_t(p) - N_t(q)\| - 2\varepsilon \geq \varepsilon$.

From Assertion 2.2 in [P],

$$\langle X - X_t, \eta \rangle_{\mathbb{M}} \geq \frac{\|N - N_t\|^2}{4} \quad (4.5)$$

with $\eta = \nabla(u - u_t)/|\nabla(u - u_t)|$ orienting the level curve Λ_t at its regular points (see also [CR]). Thus, one has

$$\int_{\Lambda_t \cap B(p, \rho_1)} \langle X - X_t, \eta \rangle \geq \frac{\rho_1 \varepsilon^2}{2}.$$

In addition, from (4.5), $\langle X - X_t, \eta \rangle$ is non negative outside the isolated points where $\nabla(u - u_t) = 0$, and so, for every compact arc $\beta \subseteq \Lambda_t$ containing $\Lambda_t \cap D(p, \rho_1)$ we have

$$\int_{\beta} \langle X - X_t, \eta \rangle \geq \frac{\rho_1 \varepsilon^2}{2}. \quad (4.6)$$

By the maximum principle, Λ_t is not compact in D . And, since Λ_t is proper on D , its two infinite branches go close to ∂D . Then there exists a connected compact part β of Λ_t , containing $\Lambda_t \cap B(p, \rho_1)$, and two arcs δ in D small enough and joining the extremities of β to ∂D . Eventually truncating by a family of horocycles, the flux formula for $X - X_t$ yields

$$0 = \int_{\beta} \langle X - X_t, -\eta \rangle + \int_{\alpha} \langle X - X_t, \nu \rangle + F_{u-u_t}(\delta \cup \delta'),$$

where α is contained in ∂D and δ' is contained in the horocycles and correctly oriented. Using (4.4) and (4.6) we obtain

$$\frac{\rho_1 \varepsilon^2}{2} \leq 2t + F_{u-u_t}(\delta \cup \delta').$$

When the length of $\delta \cup \delta'$ goes to zero, one has

$$\frac{\rho_1 \varepsilon^2}{4} \leq t.$$

Hence, if $t \leq (\rho_1 \varepsilon^2)/4$ then $\|X(p) - X_t(p)\| \leq \|N(p) - N_t(p)\| \leq 3\varepsilon$. Since ρ_1 only depends continuously on p , this gives us the desired convergence of X_t to X when t goes to zero.

After the normalization $u_t(p_0) = u(p_0)$ for a fixed $p_0 \in D$, we have that $\lim_{t \rightarrow 0} u_t|_D = u$. The convergence is uniform in relatively compact domains \tilde{D} of D and \mathcal{C}^∞ on compact sets of \tilde{D} . Hence, given a compact set $K \subseteq D$ and $\varepsilon > 0$, there exists a t small enough such that $\|u_t - u\|_{\mathcal{C}^2(K)} \leq \varepsilon$. \square

5 Entire minimal graphs.

We now establish our main result.

Theorem 5.1. *Let \mathbb{M} be a Hadamard surface with Gauss curvature bounded from above by a negative constant. Then, there exist harmonic diffeomorphisms from the complex plane onto \mathbb{M} .*

Proof. The vertical projection from a minimal surface $\Sigma \subseteq \mathbb{M} \times \mathbb{R}$ into \mathbb{M} is a harmonic map. Therefore, in order to prove the Theorem, we only need to show that there exist entire minimal graphs in $\mathbb{M} \times \mathbb{R}$ with the conformal structure of the complex plane.

Let us fix a point p_0 in a Scherk domain $D_1 \subseteq \mathbb{M}$ and also a compact disk $K_1 \subseteq D_1$. Observe that the existence of D_1 is guaranteed by Proposition 4.1.

Consider the homeomorphism h from the set $\mathbb{S}_{p_0}^1$ of unit tangent vectors at p_0 onto $\mathbb{M}(\infty)$, which maps a vector $v \in \mathbb{S}_{p_0}^1$ to the point in $\mathbb{M}(\infty)$ given by $\gamma_v(+\infty)$. Here, $\gamma_v(t)$ is the unique geodesic in \mathbb{M} with initial conditions $\gamma_v(0) = p_0$ and $\gamma'_v(0) = v$. We will measure the angle between two points $x, y \in \mathbb{M}(\infty)$ as the angle between the vectors $h^{-1}(x), h^{-1}(y) \in \mathbb{S}_{p_0}^1$.

Now, fix a sequence of positive numbers ε_n such that $\sum_{n \geq 1} \varepsilon_n < \infty$. We show the existence of an exhaustion of \mathbb{M} by Scherk domains D_n and by compact disks $K_n \subseteq D_n$ such that each K_n is contained in the interior of K_{n+1} and a sequence of minimal graphs u_n on D_n satisfying

1. $\|u_{n+1} - u_n\|_{\mathcal{C}^2(K_n)} < \varepsilon_n$,
2. the conformal modulus of the minimal annulus on $K_{i+1} - \text{int}(K_i)$ for the graph u_n is greater than one for each $1 \leq i \leq n-1$, where $\text{int}(K_i)$ denotes the interior of K_i ,
3. the angle between two consecutive vertices of the ideal polygon ∂D_n is less than $\pi/2^{n-1}$.

The third condition is clear for $n = 1$ since $p_0 \in D_1$. Thus, we assume that there exists the sequence (D_i, u_i, K_i) satisfying the three previous conditions for $1 \leq i \leq n$ and we obtain $(D_{n+1}, u_{n+1}, K_{n+1})$.

Let x, y be the vertices of a side of ∂D_n , and \mathcal{I} the arc between x and y that contains no other vertex of ∂D_n . We choose the unique point $z \in \mathcal{I}$ such that the angle between x and z agrees with the angle between y and z . Thus, the angle between x and z is less than $\pi/2^n$. Now,

from Proposition 4.1, there exists $w \in \mathcal{I}$ such that the domain bounded by the quadrilateral with vertices x, y, z, w is a Scherk domain. Moreover, the angle between two consecutive vertices is less than $\pi/2^n$.

We attach to each side of ∂D_n an ideal quadrilateral constructed as above. Then we use Proposition 4.2 and perturb all the pairs of sides of ∂D_n to obtain an ideal Scherk graph u_{n+1} on a larger domain D_{n+1} . This perturbation of the vertices can be done as small as necessary so that Conditions 1, 3 are satisfied, and also Condition 2 for $1 \leq i < n$.

Now, we use the following Lemma. We refer the reader to [CR] for its proof.

Lemma 5.1. *Every ideal Scherk surface is conformally equivalent to the complex plane.*

Hence, the minimal graph Σ of u_{n+1} is conformally the complex plane. Let $\Sigma_0 \subseteq \Sigma$ be the graph of u_{n+1} on the interior of K_n . Thus, we can choose a closed disk $\Sigma_1 \subseteq \Sigma$ containing Σ_0 in its interior such that the conformal modulus of $\Sigma_1 - \Sigma_0$ is greater than one. Then, we take K_{n+1} as the projection of Σ_1 . In addition, we can enlarge K_{n+1} , if necessary, in such a way that K_{n+1} contains $\widehat{D}_{n+1} \cap B(p_0, n)$, where \widehat{D}_{n+1} is the set of points in D_{n+1} a distance greater than 1 to its boundary and $B(p_0, n)$ the geodesic disk centered at p_0 of radius n . Thus, Condition 2 is also satisfied.

Observe now that $\mathbb{M} = \cup_{n \geq 1} D_n$. This is a straightforward consequence of Condition 3, since the set of vertices of the domains D_n is dense in $\mathbb{M}(\infty)$. In addition, from the condition between the distance of ∂K_n and ∂D_n one has that $\mathbb{M} = \cup_{n \geq 1} K_n$.

Once we have obtained the previous sequence, we can get the desired entire minimal graph. Since $u_n(p)$ is a Cauchy sequence for any $p \in \mathbb{M}$, we obtain an entire minimal graph u . On the other hand, on each compact set $K_{i+1} - \text{int}(K_i)$ the sequence u_n converges uniformly to u in the \mathcal{C}^2 -topology. Hence, the conformal modulus of the minimal graph of u on $K_{i+1} - \text{int}(K_i)$ is at least one. So, using the Grötzsch Lemma [V], the conformal type of the minimal graph of u is the complex plane. \square

We also construct harmonic diffeomorphisms from the unit disk onto \mathbb{M} by solving a Dirichlet problem at infinity.

Theorem 5.2. *Let Υ be a continuous Jordan curve in the cylinder $\mathbb{M}(\infty) \times \mathbb{R}$, which is a vertical graph. Then, there exists a unique entire minimal graph on \mathbb{M} having Υ as its asymptotic boundary. Moreover, the conformal structure of this graph is that of the unit disk.*

Proof. Let $\varphi : \mathbb{M}(\infty) \rightarrow \mathbb{R}$ be the continuous function whose graph is Υ . Let us fix a point $p_0 \in \mathbb{M}$. Consider for any unit tangent vector v at p_0 , the unique geodesic $\gamma_v(t)$ satisfying $\gamma_v(0) = p_0$ and $\gamma'_v(0) = v$, and $h : \mathbb{S}_{p_0}^1 \rightarrow \mathbb{M}(\infty)$ the homeomorphism given by $h(v) = \gamma_v(+\infty)$.

For the continuous function $\varphi \circ h : \mathbb{S}_{p_0}^1 \rightarrow \mathbb{R}$, we consider a sequence of \mathcal{C}^2 -functions $\varphi_n : \mathbb{S}_{p_0}^1 \rightarrow \mathbb{R}$ converging uniformly to $\varphi \circ h$. Then, for any positive integer n we consider the graph

on the geodesic circle centered at p_0 of radius n given by the curve $\Upsilon_n(v) = (\gamma_v(n), \varphi_n(v))$, $v \in \mathbb{S}_{p_0}^1$.

Let Σ_n be the minimal surface in $\mathbb{M} \times \mathbb{R}$ obtained as the Plateau solution with boundary Υ_n . The surface Σ_n can be seen as a graph u_n on the geodesic disk centered at p_0 and of radius n , by Rado's theorem. Since the horizontal slices are minimal surfaces, from the maximum principle, the sequence $\{u_n\}$ is uniformly bounded on compact subsets of \mathbb{M} . Thus there is a subsequence converging to an entire minimal solution $u : \mathbb{M} \rightarrow \mathbb{R}$, uniformly on compact subsets of \mathbb{M} . Let Σ be the entire minimal graph given by u .

We now prove that the asymptotic boundary of Σ is Υ . For that, observe that we only need to show that if q is a point in $\mathbb{M}(\infty) \times \mathbb{R}$ such that $q \notin \Upsilon$ then q does not belong to the asymptotic boundary of Σ .

Consider $q = (x_0, r) \in \mathbb{M}(\infty) \times \mathbb{R}$. We assume, for instance, $r > \varphi(x_0)$. Take $\varepsilon = (r - \varphi(x_0))/2 > 0$ and $v_0 = h^{-1}(x_0)$. Then, from the uniform convergence of φ_n to $\varphi \circ h$ and the continuity of φ , we can assure the existence of $\delta > 0$ and n_0 such that for all $w \in \mathbb{S}_{p_0}^1$ with $\|w - v_0\| \leq \delta$ and $n \geq n_0$

$$|\varphi_n(w) - \varphi(h(v_0))| \leq \varepsilon.$$

Let $w_1, w_2 \in \mathbb{S}_{p_0}^1$ be the unit vectors at a distance δ from v_0 . Let $\Omega \subseteq \mathbb{M}$ be the halfspace determined by the geodesic α joining the points at infinity $h(w_1)$ and $h(w_2)$ and having x_0 in $\partial_\infty \Omega$ (cf. Figure 10). From Proposition 3.1, there exists a Scherk type graph v on the halfspace Ω with boundary data $+\infty$ on α and $(r + \varphi(x_0))/2$ on $\partial_\infty \Omega$.

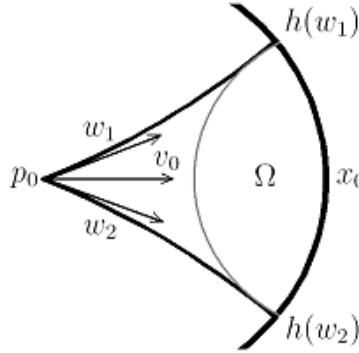


Figure 10.

From the maximum principle, $u_n \leq v$ on Ω for all $n \geq n_0$. In particular, $q = (x_0, r)$ does not belong to the asymptotic boundary of the entire graph Σ . Thus, the asymptotic boundary of Σ is Υ .

The uniqueness part of the Theorem is a straightforward consequence of the maximum principle. In addition, since the height function is harmonic and bounded for the entire minimal graph Σ , then its conformal structure must be that of the unit disk. \square

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